

ON TOTALLY PERIODIC ω -LIMIT SETS

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ABSTRACT. An ω -limit set of a continuous self-mapping of a compact metric space X is said to be totally periodic if all of its points are periodic. We say that X has the ω -FTP property provided that for each continuous self-mapping f of X , every totally periodic ω -limit set is finite. Firstly, we show that connected components of every totally periodic ω -limit set are finite. Secondly, we show in one hand, that a zero-dimensional compact metric space has the ω -FTP property, and in the other hand, for the wide class of one-dimensional continua, we prove that a hereditary locally connected X has the ω -FTP property if and only if X is completely regular. This holds in particular for X being a local dendrite with discrete set of branch points, and in particular, for a graph. For higher dimension, we show that any compact metric space X containing a free topological n -ball ($n \geq 2$) does not admit the ω -FTP property. This holds in particular, for any topological compact manifold of dimension greater than 1.

1. Introduction

Let (X, d) be a compact metric space and f be a self continuous map of X . Let \mathbb{Z}_+ and \mathbb{N} be the sets of non-negative integers and positive integers respectively. Denote by f^n the n -th iterate of f ; that is, $f^0 = \text{id}_X$: the identity and $f^n = f \circ f^{n-1}$ if $n \in \mathbb{N}$. For any $x \in X$ the subset $O_f(x) = \{f^n(x) : n \in \mathbb{Z}_+\}$ is called the f -orbit of x . A point $x \in X$ is called periodic of prime period $n \in \mathbb{N}$ if $f^n(x) = x$ and $f^i(x) \neq x$ for $1 \leq i \leq n-1$, the orbit of such point is called periodic orbit. We denote by $P(f)$ the set of periodic points and by $\text{Fix}(f)$ the set of fixed points of f . A subset $A \subset X$ is called f -invariant if $f(A) \subset A$, it is strongly f -invariant if $f(A) = A$. A p -tuple (A_0, \dots, A_{p-1}) of subsets of X is called a *periodic cycle* if $f(A_0) = A_1, f(A_1) = A_2, \dots, f(A_{p-1}) = A_0$. For a subset A of X , denote by \overline{A}

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the closure of A . We define the ω -limit set of a point $x \in X$ to be the set $\omega_f(x) = \{y \in X : \exists n_i \in \mathbb{N}, n_i \rightarrow +\infty, \lim_{i \rightarrow +\infty} d(f^{n_i}(x), y) = 0\}$.

The topological characterization of ω -limit sets of f is known only in few spaces, such as zero-dimensional spaces, this was found in [12], Theorem 13), graphs (which includes the intervals, circles), this was given by [5] and more generally, for hereditarily locally connected continuum (see below for the definition), this was given by [12]. For higher-dimensional spaces, only partial results are known; among them ω -limit sets in the square [1], [6], [3]. Apart from these results, the topological characterization of ω -limit sets is unknown. In this note, we focus on the class of ω -limit sets that are composed of periodic points. We say that $\omega_f(x)$ is *totally periodic* if $\omega_f(x) \subset P(f)$. The topological characterization of such sets is not yet been examined. The question whether a totally periodic ω -limit set is finite, is addressed.

Definition 1.1. A compact metric space X has the ω -FTP property provided that for each continuous self-mapping f of X , every totally periodic ω -limit set is finite (i.e. a periodic orbit).

The first result show that every totally periodic ω -limit set has finitely many connected components.

Theorem 1.2. *Let X be a compact metric space, f be a continuous self-mapping of X and $x \in X$. If $\omega_f(x)$ is totally periodic then $\omega_f(x)$ has finitely many connected components that form a periodic cycle.*

Corollary 1.3. *Let X be a compact metric space, f be a self continuous map of X . Let $n \geq 1$ be an integer and $x \in X$. If $\omega_f(x) \subset \text{Fix}(f^n)$ then the number of connected components of $\omega_f(x)$ divides n .*

Recall that a continuum is a compact connected metric space. We give a complete answer of the question: whether a continuum has the ω -FTP property, in the wide class of one-dimensional continua : the hereditary locally connected continua.

A space is called *degenerate* provided it has only one point; otherwise it is called *non-degenerate*. A continuum X is said to be:

- hereditarily locally connected if every subcontinuum of X is locally connected;
- completely regular if every non-degenerate subcontinuum of X has non-empty interior.
- dendrite if X is locally connected and contains no simple closed curve;

- local dendrite if every point of X has a neighborhood which is a dendrite,
- graph if X can be written as the union of finitely many arcs any two of which are either disjoint or intersect only in one or both of their end points.

Recall that if X is a local dendrite, a point $x \in X$ is called a *branch point* (resp. an *end point*) if for some neighborhood U of x , U is a dendrite and $U \setminus \{x\}$ has more than two connected components (resp. $U \setminus \{x\}$ is connected). Denote by $E(X)$ and $B(X)$ the set of all end points and branch points of X , respectively. It is known that every completely regular continuum is hereditarily locally connected ([7], Theorem 50.IV.1). Note that any graph as well as any dendrite is hereditarily locally connected. Thus every subcontinuum of a graph (resp. a dendrite) is a graph (resp. a dendrite). Any graph as well as any local dendrite with branch points discrete is completely regular. More information about hereditarily locally connected continua can be found in [10]. Our second result can be stated as follows.

Theorem 1.4. *Let X be hereditarily locally connected continuum. Then X has the ω -FTP property if and only if X is completely regular.*

In particular:

Corollary 1.5. *If X is a local dendrite with $B(X)$ discrete then X has the ω -FTP property. In particular, this holds whenever X being either:*

- a graph,
- or
- a dendrite with $B(X)$ discrete (in particular, a dendrite with $E(X)$ closed).

Remark 1. If X is a dendrite and f is a monotone continuous map of X into itself, then any totally periodic ω -limit set is finite. This follows from ([11], Theorem C).

Remark 2. In [9], Li introduced the definition of ω -scrambled set for a continuous map f of a compact metric space X into itself as follows: A subset S of X is called ω -scrambled for f if for any $x, y \in S$ with $x \neq y$:

- (i) $\omega_f(x) \setminus \omega_f(y)$ is uncountable,
- (ii) $\omega_f(x) \cap \omega_f(y)$ is nonempty,
- (iii) $\omega_f(x) \setminus P(f)$ is nonempty.

Note that condition (iii) is equivalent to say that $\omega_f(x)$ is not totally periodic. The definition of ω -scrambled set is reduced only to conditions (i) and (ii) when the space X is either a completely regular continuum or a zero-dimensional compact space; this results from Theorem 1.4 and Corollary 2.6.

In higher dimension, let $n \geq 2$ be an integer and let B_n denote the unit n -ball given by $B_n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$, where $\|\cdot\|$ is the Euclidean norm on \mathbb{R}^n , $S^{n-1} = \partial B_n$ its boundary and $\text{int}(B_n) = B_n \setminus \partial B_n$ its interior. A topological n -ball is a topological space homeomorphic to B_n . Let X be a topological space. A topological n -ball $B \subset X$ is called *free* in X if $h(\text{int}(B_n))$ is open in X where h is any homeomorphism from B_n onto B . Our third main result is the following.

Theorem 1.6. *Let X be a compact metric space containing a free topological n -ball ($n \geq 2$). Then there exists a homeomorphism of X into itself having an infinite ω -limit set consisting of fixed points. In particular, X does not admit the ω -FTP property.*

Corollary 1.7. *Every topological compact manifold of dimension ≥ 2 does not admit the ω -FTP property.*

However, the following holds:

Proposition 1.8. *Let X_1, \dots, X_n are completely regular continua and let f be the self map of $X_1 \times \dots \times X_n$ defined by $f((x_1, \dots, x_n)) = (f_1(x_1), \dots, f_n(x_n))$ where f_1, \dots, f_n are continuous self-mapping of X_i . Then every totally periodic ω -limit set of f is finite.*

The proposition 1.8 holds, for example, whenever $X_i = [0, 1]$ or $X_i = S^1$.

This paper is organized as follows. In Section 2 we prove Theorem 1.2 and Corollary 1.3. Section 3 is devoted to the one dimensional case, in particular we prove Theorem 1.4. Finally, in Section 4, we study the higher dimension by proving Theorem 1.6 and Proposition 1.8.

2. The connected component of a totally periodic ω -limit set

Let X be a compact metric space, f be a self continuous map of X and $x \in X$. We recall that the ω -limit sets possess the following basic properties:

Proposition 2.1 ([4], Lemma 2, Chapter IV). *The set $\omega_f(x)$ is a non-empty, closed and strongly invariant set.*

Proposition 2.2 ([4], Lemma 4, Chapter IV). *An ω -limit set $\omega_f(x)$ is finite if and only if it is the orbit of some periodic point.*

Lemma 2.3 ([4], Lemma 3, p. 71). *Set $L = \omega_f(x)$ and F be any non-empty proper closed subset of L . Then $F \cap \overline{f(L \setminus F)} \neq \emptyset$.*

Lemma 2.4 ([8], Proposition 3.5, p. 107). *If X is a compact metric space and totally disconnected then the clopen sets form a base for its topology.*

Lemma 2.5. *Assume that $\omega_f(x)$ is totally periodic. If $\omega_f(x)$ is totally disconnected then it is finite.*

Proof. Write $L = \omega_f(x)$. By Proposition 2.1, we have $f(L) = L$. We claim that the restriction map $f|_L : L \rightarrow L$ is a homeomorphism: Let $x, y \in L$ with period p and q respectively. If $f(x) = f(y)$ then $f^{pq}(x) = f^{pq}(y)$ hence $x = y$ then $f|_L$ is bijective. As L is compact then $f|_L$ is a homeomorphism. By hypothesis, $L \subset P(f)$, so $L = \bigcup_{n=1}^{+\infty} \text{Fix}(f^n) \cap L$.

Hence $\text{Fix}(f^p) \cap L$ has non-empty interior in L for some $p \geq 1$. By Lemma 2.4, there exists a non-empty clopen subset U of L such that $U \subset \text{Fix}(f^p)$. Denote by $F = U \cup f(U) \cup \dots \cup f^{p-1}(U)$. Then F is clopen in L and we have $f(F) = F$.

Suppose that L is infinite. One can choose U so that $F \neq L$: indeed, if $U = \{y\}$, for some $y \in U$ then $F = \{y, \dots, f^{p-1}(y)\}$ and $F \neq L$. If $U \neq \{y\}$ for some $y \in U$, we choose a non-empty clopen subset $U' \subset U \setminus \{y\}$ and so $F := U' \cup f(U') \cup \dots \cup f^{p-1}(U') \subsetneq U \cup f(U) \cup \dots \cup f^{p-1}(U)$, hence $F \neq L$.

Now by Lemma 2.3, we have $\overline{f(L \setminus F)} \cap F = f(L \setminus F) \cap F \neq \emptyset$. Hence, there is a point $z \in L \setminus F$ such that $f(z) \in F$. As $z \in P(f)$ and $f(F) = F$, thus $z \in F$, a contradiction. We conclude that L is finite. \square

As a consequence, we have the following result.

Corollary 2.6. *If X is a compact metric space and totally disconnected, then X has the ω -FTP property. In particular, Cantor space has the ω -FTP property.*

Now let X be a metric compact space, f be a self continuous map of X and assume that $L = \omega_f(x)$ is totally periodic. Denote by $Y = \overline{O_f(x)} = O_f(x) \cup L$. We have $f(Y) \subset Y$. For every $z \in Y$, $[z]$ means the connected component in Y that contains z . Let \mathcal{C} be the family of connected components of Y which form a decomposition of Y . By collapsing these components to points, we get the quotient space Y/\mathcal{C} which is a compact metric space and totally disconnected (cf. [2]). We denote by $\pi : Y \rightarrow Y/\mathcal{C}; y \mapsto [y]$ and $\tilde{f} : Y/\mathcal{C} \rightarrow Y/\mathcal{C}$ the induced map defined by: $\forall [y] \in Y/\mathcal{C}, \tilde{f}([y]) = [f(y)]$ i.e. $\tilde{f} \circ \pi = \pi \circ f$. It is plain that \tilde{f} is well defined and continuous.

Lemma 2.7. *The following assertions hold:*

- (1) *for any $n \in \mathbb{Z}_+$, $[f^n(x)] = \{f^n(x)\}$ and $[f^n(x)]$ is an isolated point in Y/\mathcal{C} .*
- (2) *If $y \in L$ then $[y] \subset L$.*
- (3) *for any $y \in Y$, $f([y]) = [f(y)]$ and $\forall n \in \mathbb{Z}_+, f^n([y]) = [f^n(y)]$.*
- (4) *for any $z \in L$, $[z]$ is a periodic point for \tilde{f} .*
- (5) *If L is infinite then $\{[y], y \in L\} = \omega_{\tilde{f}}([x])$.*

Proof. (1) If $f^n(x) \in L$ then $f^n(x) \in P(f)$ and so $Y = O_f(x)$ is finite. Therefore $f^n(x)$ is isolated in Y . If $f^n(x) \notin L$, there is $\varepsilon_1 > 0$ such that $B(f^n(x), \varepsilon_1) \cap L = \emptyset$, where $B(x_0, r)$ denotes the open ball of radius $r > 0$ and center $x_0 \in X$. In the other hand, there is $\varepsilon_2 > 0$ such that $B(f^n(x), \varepsilon_2) \cap O_f(x) = \{f^n(x)\}$ (otherwise $f^n(x) \in L$, a contradiction). So $B(f^n(x), \varepsilon) \cap Y = \{f^n(x)\}$ where $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$. Thus $f^n(x)$ is isolated in Y and hence $[f^n(x)] = \{f^n(x)\}$. As $\pi^{-1}(\{[f^n(x)]\}) = [f^n(x)] = \{f^n(x)\}$ which is open in Y , it follows that $[f^n(x)]$ is isolated in Y/\mathcal{C} .

(2) Let $y \in L$. If for some $n \in \mathbb{Z}_+$, $f^n(x) \in [y]$ then by assertion (1), $[y] = [f^n(x)] = \{f^n(x)\}$. So $y = f^n(x)$ and hence $[y] = \{y\} \subset L$. If now $O_f(x) \cap [y] = \emptyset$ then it is clear that $[y] \subset L$.

(3) By assertion (1), if $y \in O_f(x)$ then it is clear that $f([y]) = [f(y)]$. Now let $y \in L$. Since $[y]$ is the connected component of Y containing y , f is continuous and $f(Y) \subset Y$, so $f([y]) \subset Y$ is connected and contains $f(y)$, therefore, $f([y]) \subset [f(y)]$ and so $f^i([y]) \subset [f^i(y)]$, for every $i \in \mathbb{N}$. Conversely, suppose that $f([y]) \subsetneq [f(y)]$. By assertion (2) and as $f(L) = L$, $[f(y)] \subset L$ and there exists $z \in L \setminus [y]$ such that $f(z) \in [f(y)]$. Let $p, k \geq 1$ such that $f^p(y) = y$ and $f^k(z) = z$. Then we get $f^{kp}(z) = z \in [f^{kp}(y)] = [y]$, a contradiction.

(4) If $z \in L$ with period $k \geq 1$ then $\tilde{f}^k([z]) = [f^k(z)] = [z]$ and so

$[z]$ is a periodic point of \tilde{f} .

(5) Since $L = \omega_f(x) = \bigcap_{n \in \mathbb{N}} \overline{\{f^k(x) : k \geq n\}}$, we have $\pi(L) = \{[y], y \in L\} \subset \omega_{\tilde{f}}([x])$. Conversely, let $[y] \in \omega_{\tilde{f}}([x])$. If $y = f^k(x)$ for some $k \in \mathbb{Z}_+$, then $\omega_{\tilde{f}}([x]) = \omega_{\tilde{f}}([y])$. Now by (1), $[y]$ is isolated in Y/\mathcal{C} and $\{f^k(x)\} = [y]$. Hence $[y] = \tilde{f}^{k+p}([x])$ for infinitely many p . Therefore $[y]$ is \tilde{f} -periodic and so $f^k(x)$ is f -periodic point. Thus L is finite, a contradiction. We conclude that $y \in L$ and therefore $\omega_{\tilde{f}}([x]) = \{[y], y \in L\}$. \square

Proof of Theorem 1.2. If L is finite, the proof is clear. Assume that L is infinite. As Y/\mathcal{C} is compact and totally disconnected metric space, and $\omega_f(x) \subset P(f)$, then $\omega_{\tilde{f}}([x])$ is totally disconnected and by Lemma 2.7, (4) and (5), $\omega_{\tilde{f}}([x]) \subset P(\tilde{f})$. Therefore by Lemma 2.5, $\omega_{\tilde{f}}([x])$ is finite i.e. $\omega_{\tilde{f}}([x]) = \{[z], \tilde{f}([z]), \dots, \tilde{f}^{k-1}([z])\}$ (for some $z \in Y$ and $k \geq 1$). As $\tilde{f}^i([z]) = f^i([z])$ (Lemma 2.7, (3)) we conclude that L has a finite number of connected components: $[z], f([z]), \dots, f^{k-1}([z])$ that form a periodic cycle for f . The proof is complete. \square

Proof of Corollary 1.3. As $\omega_f(x) = \omega_{f^n}(x) \cup \omega_{f^n}(f(x)) \dots \cup \omega_{f^n}(f^{n-1}(x))$ then for each $0 \leq i \leq n-1$, $\omega_{f^n}(f^i(x)) \subset \text{Fix}(f^n)$. By Theorem 1.2, $\omega_{f^n}(f^i(x))$ has finitely many connected components C_1, \dots, C_p , so each C_k , ($1 \leq k \leq p$) is open in $\omega_{f^n}(f^i(x))$. Suppose that $\omega_{f^n}(f^i(x))$ is not connected, then $p \geq 2$ and by applying Lemma 2.3, there exists $z \in \omega_{f^n}(f^i(x)) \setminus C_1$ such that $f^n(z) \in C_1$, a contradiction since $f^n(z) = z$. We conclude that for each $1 \leq i \leq n$, $\omega_{f^n}(f^i(x))$ is connected and therefore the number l of connected components of $\omega_f(x)$ is at most n . As the connected components of $\omega_f(x)$ form a periodic cycle for f then l divides n . \square

3. The ω -FTP property in one-dimension

3.1. The example. We are going to construct a continuous map F_0 on a dendrite D_0 admitting an ω -limit set which is an arc formed only by fixed points.

(a) First, we define the space D_0 as a subset of the plane in the following way:

Let $n \in \mathbb{N}$. Denote by $S_n = \{\frac{i}{2^n} : 1 \leq i \leq 2^n - 1\}$. For $n = 1$, we let $\Omega_1 = \{\frac{1}{2}\}$ and for each $n > 1$, we let

$$\Omega_n = S_n \setminus \bigcup_{i=1}^{n-1} \Omega_i.$$

For example, $\Omega_2 = \{\frac{1}{4}, \frac{3}{4}\}$, $\Omega_3 = \{\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}\}$. For each $n \in \mathbb{N}$, the cardinal of Ω_n is 2^{n-1} . We define now a dense sequence $(a_n)_{n \in \mathbb{N}}$ in $[0, 1]$ as follow:

For $n = 1$, we let $a_1 = \frac{1}{2}$. For each $n \in \mathbb{N}$, set

- $\Omega_{2n} = \{a_i : 2^{2n-1} \leq i \leq 2^{2n} - 1\}$ given in this order: $a_{2^{2n-1}} < \dots < a_{2^{2n}-1}$.
- $\Omega_{2n+1} = \{a_i : 2^{2n} \leq i \leq 2^{2n+1} - 1\}$ given in this order: $a_{2^{2n}} > \dots > a_{2^{2n+1}-1}$.

For example, $a_2 = \frac{1}{4}$, $a_3 = \frac{3}{4}$, $a_4 = \frac{7}{8}$, $a_5 = \frac{5}{8}$, $a_6 = \frac{3}{8}$, $a_7 = \frac{1}{8}$.

For any $n, k \in \mathbb{N}$ such that $a_k \in \Omega_n$, we let $I_k = \{a_k\} \times [0, \frac{1}{n}]$. The dendrite D_0 is then given by:

$$D_0 := ([0, 1] \times \{0\}) \cup \bigcup_{k \in \mathbb{N}} I_k.$$

(b) Second we construct the map F_0 as follows: For any $k \in \mathbb{N}$ such that $a_k \in \Omega_n$, write $I_k = \{a_k\} \times [0, \frac{1}{n}] = [A_k, B_k]$, where $A_k = (a_k, 0)$, $B_k = (a_k, \frac{1}{n})$. We let $C_k = (a_k, \frac{2}{n})$. Then the map F_0 is defined inductively as follows:

- $[0, 1] \times \{0\} = \text{Fix}(F_0)$.
- $(F_0)|_{[A_k, C_k]} : [A_k, C_k] \rightarrow [A_k, A_{k+1}]$ be a homeomorphism that fixes the point A_k and maps linearly $[A_k, C_k]$ to $[A_k, A_{k+1}]$,
- $(F_0)|_{[C_k, B_k]} : [C_k, B_k] \rightarrow [A_{k+1}, B_{k+1}]$ be a homeomorphism that maps linearly $[C_k, B_k]$ to $[A_{k+1}, B_{k+1}]$ and sends C_k to A_{k+1} and B_k to B_{k+1} .

The map F_0 is continuous on D_0 : Let $k \in \mathbb{N}$ and $M \in I_k \setminus A_k$. Then there is a sub-arc of I_k containing M , which is open in D_0 . Hence F_0 is continuous at M since F_0 is a homeomorphism on I_k . Let $M = (x, 0) \in [0, 1] \times \{0\}$ and (M_n) be a sequence of points in D_0 that converges to M . If M_n lies eventually in $[0, 1] \times \{0\}$ then $F_0(M_n) = M_n$ and so $\lim_{n \rightarrow +\infty} F_0(M_n) = M$. If M_n lies eventually in $I_{k(n)} \setminus A_{k(n)}$ then we distinguish three subcases:

- Case 1. M_n lies finitely many times in each $I_{k(n)} \setminus A_{k(n)}$. In this case, $\lim_{n \rightarrow +\infty} k(n) = +\infty$. Since $\lim_{n \rightarrow +\infty} \text{diam}(I_{k(n)}) = 0$, we obtain $\lim_{n \rightarrow +\infty} A_{k(n)} = M$. We also have $\lim_{n \rightarrow +\infty} \text{diam}([A_{k(n)}, A_{k(n)+1}]) = 0$ and for eventually n , $F_0(M_n) \in [A_{k(n)}, A_{k(n)+1}] \cup I_{k(n)+1}$. Therefore $\lim_{n \rightarrow +\infty} F_0(M_n) = M$.

- Case 2. M_n lies in a finite union of I_k (i.e. $k(n)$ is bounded). In this case, then there exists $k_0 \in \mathbb{N}$ such that M_n lies eventually in I_{k_0} and $A_{k_0} = M$. By the continuity of the restricted map $(F_0)_{|I_{k_0}}$, we get $\lim_{n \rightarrow +\infty} F_0(M_n) = M$.
- Case 3. There exists $k_0 \in \mathbb{N}$ such that M_n lies infinitely many times in I_{k_0} and $M_n \notin I_{k_0}$ for infinitely many times. In this case, the sequence $(M_n)_n$ has two subsequences $(M_{\alpha(n)})_n$ and $(M_{\beta(n)})_n$ such that $\mathbb{N} = \{\alpha(n) : n \in \mathbb{N}\} \cup \{\beta(n) : n \in \mathbb{N}\}$ where $(M_{\alpha(n)})$ satisfies case 1 and $(M_{\beta(n)})$ satisfies the case 2. So $\lim_{n \rightarrow +\infty} F_0(M_n) = M$.

In result, F_0 is continuous on $[0, 1] \times \{0\}$ and so F_0 is continuous on the whole dendrite D_0 . The F_0 -orbit of the point B_1 is then $O_{F_0}(B_1) = \{B_k : k \in \mathbb{N}\}$. Hence $\omega_{F_0}(B_1) = [0, 1] \times \{0\} = \text{Fix}(F_0)$.

3.2. Proof of Theorem 1.4

• Assume that X is completely regular and let $L = \omega_f(x) \subset P(f)$. We claim that L is totally disconnected. Indeed, otherwise L contains a non-degenerate connected component C necessarily with non-empty interior in X . In particular L is infinite and there exists $n \in \mathbb{N}$ such that $f^n(x) \in L$. Thus $f^n(x)$ is a periodic point. As $\omega_f(x) = \omega_f(f^n(x))$, then L is finite, a contradiction. Now by Lemma 2.5, L itself is finite.

• Conversely, suppose that X is not completely regular.

(a) First we prove that X contains a homeomorphic copy of the dendrite D_0 (see the Example 3.1 above). Indeed, there is a non-degenerate locally connected subcontinuum Y of empty interior. As Y itself is arcwise connected, it contains an arc $[a, b]$. Take $c \in (a, b)$ and $\varepsilon > 0$. As X is locally arcwise connected, there is an arcwise connected neighborhood U of c in X with diameter less than ε (see [10], Theorem 8.25, p.131). As Y has empty interior in X , there exists $d \in U \setminus [a, b]$ and so an arc, say $[c, d]$ in U , joining c and d . Let $r \in [c, d[$ be such that the sub-arc $[c, r]$ of $[c, d]$ intersects $[a, b]$ only in the point r . Hence we have proved that for each $\varepsilon > 0$ and $c \in (a, b)$, there is $r \in [a, b]$ with $d(r, c) < \varepsilon$ and an arc I with r as one of its endpoints such that $I \cap [a, b] = \{r\}$. So if we take two distinct points $c, c' \in (a, b)$, we can find two disjoint arcs one has c as one of its endpoints and the

other has c' as one of its endpoints and each one of them intersects $[a, b]$ only in c and c' respectively. We conclude the existence of a dense sequence $(r_n)_n$ of points in $[a, b]$ and a pairwise disjoint arcs $(I_n)_n$, each of them has r_n as one of its endpoints with $I_n \cap [a, b] = \{r_n\}$. Since X is hereditarily locally connected, then by ([7], Theorem 50.IV.9), the arcs $(I_n)_n$ is a null family; i.e. for any $\varepsilon > 0$, only finitely many I_n have diameter $> \varepsilon$. Therefore $\lim_{n \rightarrow +\infty} \text{diam}(I_n) = 0$. We conclude that the set $Z := [a, b] \cup \bigcup_{n \in \mathbb{N}} I_n$ is homeomorphic to the dendrite D_0 .

(b) Second, we need the following proposition.

Proposition 3.1 ([12], proposition 11). *Let X be a hereditarily locally connected continuum and let Y be a locally connected continuum. Then any continuous map $f : M \rightarrow Y$ defined on a closed subset M of X can be extended to a continuous map $F : X \rightarrow Y$ defined on X .*

Now, we complete the proof of Theorem 1.4 as follows. From the Example 3.1, there exist a point $x \in Z$ and a continuous map $F : Z \rightarrow Z$ defined on Z such that $\omega_F(x) = [a, b] = \text{Fix}F$. So, by Proposition 3.1, the map F can be extended to a continuous map \tilde{F} defined on X into X with $\omega_{\tilde{F}}(x) = [a, b] \subset \text{Fix}\tilde{F}$. Therefore X does not admit the ω -FTP property. \square

4. The ω -FTP property in higher dimension

Proposition 4.1. *For each integer $n \geq 2$, there exists a homeomorphism f of B_n onto itself such that $S^{n-1} \cup \{0\} \subset \text{Fix}f$ and $\omega_f(x)$ is a circle included in S^{n-1} , for some $x \in B_n \setminus (S^{n-1} \cup \{0\})$.*

Proof. We shall construct such homeomorphism f by induction on n .

• The case $n = 2$. In this case B_2 is the unit disk, denoted by $D = \{z \in \mathbb{C} : |z| \leq 1\}$. Let $z_0 = r_0 e^{i \frac{1}{1-r_0^2}} \in D$ where $0 < r_0 < 1$ is a real. We shall construct a C^∞ -diffeomorphism f on D such that $\omega_f(z_0) = \partial D = S^1$ with $\text{Fix}(f) = \partial D \cup \{0\}$. Such a diffeomorphism will be the time-one map of a C^∞ -flow on D . Let X be the vector fields defined on D by

$$X(z) = \begin{cases} e^{\frac{i}{1-|z|^2}} (|z|^2 + 2i(1-|z|^2)|z|^4) \varphi(z), & \text{if } |z| < 1 \\ 0, & \text{if } |z| = 1 \end{cases}$$

where

$$\varphi(z) = \begin{cases} e^{\frac{-1}{1-|z|^2}}, & \text{if } |z| < 1 \\ 0, & \text{if } |z| = 1 \end{cases}$$

The map φ is C^∞ on D . Indeed, set $h(x) = e^{\frac{-1}{1-x^2}}$, $x \in [0, 1[$.

The derivative of h of order n , ($n \in \mathbb{Z}_+$) is: $h^{(n)}(x) = P_n(x) \frac{e^{\frac{-1}{1-x^2}}}{(1-x^2)^{2n}}$, where P_n is a polynomial function. Hence $\lim_{x \rightarrow 1} h^{(n)}(x) = 0$ and so h is C^∞ on 1 . We conclude that φ is C^∞ on D and so on X . The set of singular points of X is $\{0\} \cup S^1$. Let $(\phi_t)_{t \in \mathbb{R}}$ denote the flow on D associated to X . Write $\phi_t(z_0) := z(t) = r(t)e^{i\theta(t)}$, $t \in \mathbb{R}$ where $z(0) = z_0$, be the solution of the differential equation $z'(t) = X(z(t))$ with $z(0) = z_0$. Then r is a solution of the real differential equation $r'(t) = r^2(t)\varphi(r(t))$ with $r(0) = r_0$ and $\theta(t) = \frac{1}{1-r^2(t)}$. The function r satisfies $\lim_{t \rightarrow +\infty} r(t) = 1$, $\lim_{t \rightarrow -\infty} r(t) = 0$. The function θ is increasing

and satisfies $\lim_{t \rightarrow +\infty} \theta(t) = +\infty$ and $\lim_{|t| \rightarrow +\infty} \theta'(t) = \lim_{|t| \rightarrow +\infty} \frac{2r^3\varphi(r(t))}{(1-r^2(t))^2} = 0$.

We let $f = \phi_1$ be the time one map of the flow. Then f is a C^∞ -diffeomorphism of D . Let us show that $\omega_f(z_0) = S^1$. One has $f^n(z_0) = \phi_n(z_0)$, so $\omega_f(z_0) \subset S^1$. Conversely, if $z \in S^1$, there is a sequence $(t_n)_n \subset \mathbb{R}$ with $t_n \rightarrow +\infty$ such that $\phi_{t_n}(z_0) \rightarrow z$. As $\lim_{n \rightarrow +\infty} r(n) = 1$ and $\lim_{n \rightarrow +\infty} \theta(n+1) - \theta(n) = 0$, it follows that $\lim_{n \rightarrow +\infty} |f^{[t_n]}(z_0) - \phi_{t_n}(z_0)| = 0$ where $[t_n]$ is the integer part of t_n . Therefore $\lim_{n \rightarrow +\infty} f^{[t_n]}(z_0) = z$ and so $S^1 \subset \omega_f(z_0)$. We conclude that $\omega_f(z_0) = S^1$.

• Suppose that for $n \geq 2$, there exists a homeomorphism f_n of B_n onto itself satisfying the required properties. Let f_{n+1} denote the map given by:

$$f_{n+1}(x) = \begin{cases} \left(\sqrt{1-x_{n+1}^2} f_n\left(\frac{1}{\sqrt{1-x_{n+1}^2}}(x_1, x_2, \dots, x_n)\right), x_{n+1} \right), & \text{if } x_{n+1} \neq \pm 1 \\ (0, \dots, \pm 1), & \text{if } x_{n+1} = \pm 1 \end{cases}$$

where $x = (x_1, x_2, \dots, x_n, x_{n+1}) \in B_{n+1}$. It is plain that f_{n+1} is continuous and bijective, hence f_{n+1} is a homeomorphism of B_{n+1} onto itself. In addition, $S^n \cup \{0\} \subset \text{Fix} f_{n+1}$. For any $x \in (x_1, x_2, \dots, x_n, x_{n+1}) \in$

B_{n+1} , we have

$$\omega_{f_{n+1}}(x) = \sqrt{1 - x_{n+1}^2} \omega_{f_n} \left(\frac{1}{\sqrt{1 - x_{n+1}^2}}(x_1, x_2, \dots, x_n) \right) \times \{x_{n+1}\}.$$

We conclude that there is a point $x_0 \in B_{n+1} \setminus (S^n \cup \{0\})$ such that $\omega_{f_{n+1}}(x_0) \subset S^n$ and is homeomorphic to the circle S^1 . \square

Proof of Theorem 1.6. Let V be a free topological n -ball in X . Its boundary ∂V is homeomorphic to the sphere S^{n-1} . By Proposition 4.1, there exists a homeomorphism $f : V \rightarrow V$ and a point $c \in V$ such that $\partial V \cup \{c\} \subset \text{Fix} f$ and for some point $x \in V \setminus \{c\}$, $\omega_f(x) \subset \partial V$ is homeomorphic to the circle S^1 . In this way, the map f can be easily extended to a continuous map, say \tilde{f} defined on the whole space X by fixing all points in $X \setminus V$. Hence \tilde{f} is homeomorphism of X onto itself, where $\omega_{\tilde{f}}(x)$ is infinite and consisting of fixed points. We conclude that X does not admit the ω -FTP property. \square

Proof of Proposition 1.8. Let $(x_1, \dots, x_n) \in X_1 \times \dots \times X_n$ and assume that $\omega_f((x_1, \dots, x_n)) \subset P(f)$. Let $i \in \{1, \dots, n\}$ and $a_i \in \omega_{f_i}(x_i)$, then there exists a sequence $(n_k^{(i)})_k \rightarrow +\infty$ such that $f_i^{n_k^{(i)}}(x_i) \rightarrow a_i$. Now the sequence $(f^{n_k^{(i)}}((x_1, \dots, x_n)))_k = (f_1^{n_k^{(i)}}(x_1), \dots, f_n^{n_k^{(i)}}(x_n))_k$ has a subsequence that converges to a point $(b_1^{(i)}, \dots, a_i, \dots, b_n^{(i)}) \in P(f)$. As $P(f) \subset P(f_1) \times \dots \times P(f_n)$ then $a_i \in P(f_i)$. We conclude that $\omega_{f_i}(x_i) \subset P(f_i)$. By Theorem 1.4, the $\omega_{f_i}(x_i)$ ($1 \leq i \leq n$) are finite. As $\omega_f(x_1, \dots, x_n) \subset \omega_{f_1}(x_1) \times \dots \times \omega_{f_n}(x_n)$, hence $\omega_f(x_1, \dots, x_n)$ is finite. \square

Recall that a Peano continuum can be defined as a continuous image of $[0, 1]$. In view of Theorem 1.4, one may be tempted to ask the following question.

Question. Is it true that a Peano continuum has the ω -FTP property if and only if it is completely regular ?

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